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SKEW-SELFADJOINT FORM FOR

SYSTEMS OF CONSERVATION LAWS

Eitan Tadmor

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SKEW-SELFADJOINT FORM FOR

SYSTEMS OF CONSERVATION LAWS

Eitan Tadmor
Institute For Computer Applications in Science and Engineering

ABSTRACT

We study hyperbolic systems of conservation laws augumented with an entropy inequality. We show that such systems can be written in a (quasilinear) skew-selfadjoint form. Centered differencing of such a form under the smooth regime, ends up with a systematic recipe for constructing quasiconservative schemes where the global entropy conservation is recovered. Employing the above formulation in bounded regions under the nonsmooth regime as well, we further conclude a local entropy decay estimate. Examples of the shallow-water and the full gasdynamics equations are explicitly treated.

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1. Introduction

We consider the hyperbolic system of conservation laws

(1.1a) $$\frac{\partial}{\partial t} \underset{\sim}{u} + \sum_{k=1}^{d} \frac{\partial}{\partial x_k} [\underset{\sim}{f}^{(k)}] = 0, \quad (\bar{x},t)\varepsilon \mathscr{R}^d \times [0,\infty) \ ;$$

here the N-dimensional vector of unknowns $\underset{\sim}{u} \equiv \underset{\sim}{u}(\bar{x},t)=(u_1,\ldots,u_N)^T$ is to be found subject to the initial state

$$\underset{\sim}{u}(\bar{x},t=0) = \underset{\sim}{u}_0(\bar{x}), \quad \bar{x}\varepsilon \mathscr{R}^d ,$$

and $\underset{\sim}{f}^{(k)} \equiv \underset{\sim}{f}^{(k)}(\underset{\sim}{u})=(f_1^{(k)},\ldots,f_N^{(k)})^T$ are smooth nonlinear (flux) mappings from $\mathscr{D} \subset \mathscr{R}^N$ to $\mathscr{R}^N$.

Independently of the initial smoothness of $\underset{\sim}{u}_0(\bar{x})$, solutions of (1.1) may develop singularities at a finite time after which one must admit <u>weak</u> solutions, i.e., those derived directly from the original integral conservation relations, see [9]. Admitting weak solutions however, sacrifices uniqueness which we are then trying to recover by appealing to a unique physically relevant solution; the later being identified as, roughly speaking, a stable limit of a vanishing dissipativity mechanism.

To this end we introduce a generalized <u>Entropy function</u> - a smooth <u>convex</u> mapping $U(\underset{\sim}{u})$ from $\mathscr{D} \subset \mathscr{R}^N$ to $\mathscr{R}$ augumented with entropy fluxes $F^{(k)}(\underset{\sim}{u}):\mathscr{D} \to \mathscr{R}$ such that

(1.2a) $$U_{\underset{\sim}{u}}^T A_k = F_{\underset{\sim}{u}}^{(k)T}, \quad A_k \equiv \frac{\partial}{\partial u} [\underset{\sim}{f}^{(k)}], \ k = 1, 2, \ldots d.$$

Multiplying (1.1a) by $U_{\underset{\sim}{u}}^T$ and employing (1.2a) one finds that <u>under the smooth regime</u> we have, on top of (1.1a), the additional conservation of entropy

$$\frac{\partial}{\partial t} U + \sum_{k=1}^{d} \frac{\partial}{\partial x_k} [F^{(k)}] = 0 .$$

Taking into account the <u>nonsmooth regime</u> as well, following Lax [10] we postulate as an admissibility criterion the

Entropy inequality - we have, in the sense of distributions,

$$\frac{\partial}{\partial t} U + \sum_{k=1}^{d} \frac{\partial}{\partial x_k} [F^{(k)}] \le 0 . \tag{1.2b}$$

Having a (weakly) nonpositive quantity on the L.H.S. of (1.2b) exactly recovers the existence of vanishing dissipativity in our system, and it in fact measures the rate in which entropy dissipates across shock distontinuities. Though the entropy inequality is not a powerful enough criterion to rule out all the (inadmissible) unstable weak solutions, it is equivalent in the small to Lax's shock conditions and as such is necessary for stability. For a detailed account of the role of entropy admissibility criteria with respect to stability of weak solutions we refer to DiPerna [1] and the references therein.

The existence of an entropy function turns out to have also a decisive role in studying stability under the smooth regime; indeed, upon multiplying (1.1a) on the left by the Hessian $U_{\underset{\sim}{u}\underset{\sim}{u}} > 0$, one ends up with a (quasilinear) symmetric hyperbolic system for which the classical (local) well-posedness theory prevail, see Friedrichs and Lax [2].

As was pointed out by Mock (cf.[5]), the existence of an entropy function further implies that system (1.1a) can be symmetrized with respect to a new variable $\underset{\sim}{v}$, $\underset{\sim}{v} = U_{\underset{\sim}{u}}$, i.e., (1.1a) can be rewritten as

$$\tilde{A}_0 \frac{\partial}{\partial t} \underset{\sim}{v} + \sum_{k=1}^{d} \tilde{A}_k \frac{\partial}{\partial x_k} \underset{\sim}{v} = 0 \tag{1.3a}$$

with symmetric

$$\tilde{A}_0 \equiv \tilde{A}_0(\underset{\sim}{v}) = \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{v}} > 0 , \quad \tilde{A}_k \equiv \tilde{A}_k(\underset{\sim}{v}) = \frac{\partial}{\partial \underset{\sim}{v}} [\underset{\sim}{f}^{(k)}] . \tag{1.3b}$$

Godunov has previously proved that the converse is also true and hence the symmetrizable systems are exactly those equipped with a convex extension -- an entropy. Noting that (see (1.2a))

$$\tilde{A}_k = A_k \tilde{A}_0 , \tag{1.3c}$$

the above mentioned Friedrichs-Lax (matrix) symmetrization upon multiplying by $U_{\underset{\sim}{u}\underset{\sim}{u}} \equiv \tilde{A}_0^{-1}$ follows -- indeed $\tilde{A}_0^{-1} A_k$ are symmetric; the former (variable) symmetrization in (1.3a) is more fundamental however, as it preserves strong <u>as well as weak</u> solutions of the original system (1.1a).

Our starting point is closely related to the symmetrization in (1.3a) in the sense that we too are considering the symmetrizing variables $\underset{\sim}{v} = U_{\underset{\sim}{u}}$ as the primary dependent unknowns rather than the standard conserved quantity $\underset{\sim}{u}$. Motivated by the gasdynamics equations we show in Section 2 that under the further assumption of the entropy $U(\underset{\sim}{u})$ and the fluxes $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$ being <u>homogeneous,</u> one can in particular rewrite system (1.3a) in a (quasilinear $\underset{\sim}{v}$-dependent) <u>skew-selfadjoint</u> from -- see (2.5) below; as explained later, <u>the homogeneity restriction can be essentially removed.</u> Such a skew-selfadjoint form seems to be of independent applicable interest under both the smooth as well as the nonsmooth regimes.

As an example for the former we show, in Section 3, that the additional entropy conservation

$$\int_{\mathscr{R}^d} U(\cdot,t)d\bar{x} = \int_{\mathscr{R}^d} U(\cdot,t=0)d\bar{x} , \tag{1.4}$$

can be directly recovered from the skew-selfadjoint conservative equations using nothing else but integration by parts. The above derivation lends itself to discrete approximations based on centered differencing of the skew-selfadjoint form of the equations, e.g., finite differences, pseudospectral differencing [7], [3, Chapter 14]. Thus the above description provides us with a <u>systematic procedure</u> of discretizing the nonlinear equations while maintaining their quasiconservative properties. In other words, the quasiconservative schemes obtained by such a recipe enjoy the additional conservation property induced by the corresponding differential one, namely that of the global entropy, see (1.4). Skew-symmetric differencing such as the above suggests itself, for example, as a remedy to nonlinear instabilities (cf. long-term weather prediction integration in [4] and the references therein), or in connection with the aliasing-free pseudospectral skew-differencing proposed by Kreiss and Oliger [7] to stabilize discretizations of the linear problem. The fact that it is the entropy rather than an L_2-equivalent quantity which is conserved in the recipe above is exactly the reason allowing us to treat the nonlinear problem as well; we clarify this point as well as other aspects of discretizing the pure initial-value problem in Section 3.

In Section 4 we extend our discussion to the nonsmooth regime, treating the entropy decay in bounded regions. For the latter to occur we must require an entropy outflux at the boundaries; employing the above mentioned skew-selfadjoint form, we are able to express such a requirement as a maximal dissipativity-like condition which leads to our main result, see theorem 4.1 below,

$$(1.5) \qquad \int_{|\bar{x}|\le M} U(\cdot,t)\,d\bar{x} \le \int_{|\bar{x}|\le M+st} U(\cdot,t=0)\,d\bar{x},$$

thus sharpening the global estimate we had in (1.4) to be of a local type. The above estimate is indeed sharp -- in the case of first order homogeneous fluxes $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$ for example (which includes the hydrodynamic equations), the magnitude of the speed propagation, $|s|$, is found to coinside with the one induced by the 'nonlinear symbol' $\sum_{k=1}^{d} \omega_k A_k$. The speed orientation sgn(s) is either positive or negative depending on whether the order of U-homogeneity is bigger or smaller than 1 -- see Section 4 below for a fuller account on the distinction between these two subcases.

The homogeneity assumption we made is not met in many cases of physical interest, the shallow-water and magnetohydrodynamic equations are just two examples. In Section 5 we show how a skew-selfadjoint form still can be derived, extending our procedure above to nonhomogeneous framework such as, for example, the one of the shallow-water equations.

We close by remarking that the skew-selfadjoint form obtained in the $\underset{\sim}{v}$-variables, can be also viewed as if the system in the original $\underset{\sim}{u}$-variables being skew-selfadjoint with respect to appropriately chosen inner product -- see (3.5b) below. This point of view suggests itself for various applications; details will be given elsewhere.

2. A Skew-Selfadjoint Form

Motivated by the gasdynamics equations we start with

Homogeneity assumption - the entropy function $U(\underset{\sim}{u})$ and the fluxes $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$ are homogeneous of order η_0 and η_k, respectively.

Indeed, for the (polytropic) gasdynamics equations we let $\underset{\sim}{u} = (\rho,\underset{\sim}{m},E)^T$ stand for the unknown vector of density ρ, momentum $\underset{\sim}{m}=(m_1,m_2,m_3)^T$ and energy E with the corresponding fluxes (here $[\underset{\sim}{e}^{(k)}]_j=\delta_{kj}$)

$$\underset{\sim}{f}^{(k)} = (m_k, \frac{m_k}{\rho}\underset{\sim}{m} + p\,\underset{\sim}{e}^{(k)}, (E+p)\underset{\sim}{m}/\rho)^T , \quad k = 1, 2, 3, \tag{2.1a}$$

the pressure p is given by $p = (\gamma-1)[E-|\underset{\sim}{m}|^2/2\rho]$. Using any of the following α-parameter family of generalized entropies [5] (here the adiabatic exponent $\gamma>1$),

$$U(\underset{\sim}{u}) = -(p\rho^{\alpha})^{1/\alpha+\gamma} , \quad \alpha > 0 , \tag{2.1b}$$

the homogeneity assumption is fulfilled with

$$\eta_0 = \frac{\alpha+1}{\alpha+\gamma} , \quad \eta_1 = \eta_2 = \eta_3 = 1 . \tag{2.1c}$$

The entropy homogeneity implies that $U_{\underset{\sim}{u}}(\underset{\sim}{u})$ is homogeneous of order η_0-1 in its argument and hence $\underset{\sim}{u} \equiv \underset{\sim}{u}(\underset{\sim}{v})$ is homogeneous of order $\frac{1}{\eta_0-1}$ in the dependent (symmetrizing) variable $\underset{\sim}{v} = U_{\underset{\sim}{u}}^{(1)}$. This in turn induces the homogeneity of $\underset{\sim}{f}^{(k)}(\underset{\sim}{v}) \equiv \underset{\sim}{f}^{(k)}(\underset{\sim}{u}(\underset{\sim}{v}))$ of order $\frac{\eta_k}{\eta_0-1}$, respectively. We now

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(1) By the convexity of $U(\underset{\sim}{u})$, the relation $\underset{\sim}{u}=\underset{\sim}{u}(\underset{\sim}{v}=U_{\underset{\sim}{u}})$ is well-defined. Also, η_0 is necessarily different from 1 - see (2.3).

recall the $\tilde{A}$'s notations in (1.3b) and use Euler's identity to find

$$\frac{1}{\eta_0 - 1}\cdot \underset{\sim}{u} = \tilde{A}_0 \underset{\sim}{v} \ , \qquad \frac{\eta_k}{\eta_0 - 1}\cdot \underset{\sim}{f}^{(k)} = \tilde{A}_k \underset{\sim}{v} \ , \tag{2.2}$$

where all quantitities depend on the dummy argument $\underset{\sim}{v}$. We use Euler's identity once more - this time as a second λ-differentation of the (assummed homogeneous) entropy $U(\lambda \underset{\sim}{u}) = \lambda^{\eta_0} U(\underset{\sim}{u})$ at $\lambda = 1$, which gives

$$\underset{\sim}{u}^* U_{\underset{\sim\sim}{uu}} \underset{\sim}{u} = \eta_0(\eta_0 - 1)\cdot U(\underset{\sim}{u}) \ . \tag{2.3}$$

For the assummed U-convexity to hold therefore, U must be one-signed - either negative or positive with $0 < \eta_0 < 1$ or $\eta_0 > 1$, respectively; in either case $\eta_0 \neq 0$. Finally with the help of (2.2) we rewrite the temporal and spatial derivatives appearing in (1.3a) in the form

$$\frac{\partial}{\partial t} u = \frac{\eta_0 - 1}{\eta_0}\cdot \underset{\sim}{u}_t + \frac{1}{\eta_0}\cdot \partial_t(\underset{\sim}{u}) = \frac{\eta_0 - 1}{\eta_0}\cdot\left[\tilde{A}_0 \underset{\sim}{v}_t + \partial_t(\tilde{A}_0 \underset{\sim}{v})\right] , \tag{2.4a}$$

$$\frac{\partial}{\partial x_k} \underset{\sim}{f}^{(k)} = \frac{\eta_0 - 1}{\eta_0 + \eta_k - 1}\cdot \underset{\sim}{f}^{(k)}_{x_k} + \frac{\eta_k}{\eta_0 + \eta_k - 1}\cdot \partial_{x_k}\left(\underset{\sim}{f}^{(k)}\right) = \frac{\eta_0 - 1}{\eta_0 + \eta_k - 1}\cdot\left[\tilde{A}_k \underset{\sim}{v}_{x_k} + \partial_{x_k}(\tilde{A}_k \underset{\sim}{v})\right] . \tag{2.4b}$$

Thus we have shown

THEOREM 2.1. System (1.1a) can be rewritten in the form

$$\mathscr{L}(\underset{\sim}{v})\underset{\sim}{v} = 0 \ , \qquad \underset{\sim}{v} = U_{\underset{\sim}{u}} \ . \tag{2.5a}$$

Here the (quasilinear) skew-selfadjoint operator $\mathscr{L} \equiv \mathscr{L}(\underset{\sim}{v})$ is given by

$$\mathscr{L} \equiv \tilde{B}_0 \frac{\partial}{\partial t} + \frac{\partial}{\partial t}(\tilde{B}_0 \cdot) + \sum_{k=1}^{d} \tilde{B}_k \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k}(\tilde{B}_k \cdot) \ , \tag{2.5b}$$

in terms of the symmetric matrix coefficients

$$\begin{aligned} \tilde{B}_0 &\equiv \tilde{B}_0(\underset{\sim}{v}) = \tilde{A}_0(\underset{\sim}{v}) > 0 \\ \tilde{B}_k &\equiv \tilde{B}_k(\underset{\sim}{v}) = \frac{\eta_0}{\eta_0 + \eta_k - 1}\cdot \tilde{A}_k(\underset{\sim}{v}) \ , \qquad \eta_0 + \eta_k - 1 \neq 0 \ . \end{aligned} \tag{2.5c}$$

The formulation (2.5) is in general not conservative unless complemented with the homogeneity property.

In the case of first order homogeneous fluxes, $\eta_k=1$, one can simply identify the $\tilde{B}$'s with the $\tilde{A}$'s. This is the above mentioned hydrodynamic case for which an explicit representation of the $\tilde{A}$'s can be found in [5, Section 2]. As can be easily verified we note that the only choice for an homogeneous generalized entropy density given as a function of the thermodynamic one, $S = c_v \ln(p\rho^{-\gamma})$, is minus the exponential function which indeed leads to (2.1b) - see [5, Section 2].

3. Entropy Conservation in the Cauchy Problem

We consider the quasilinear system (2.5)

$$P(\underset{\sim}{v})\underset{\sim}{v} + Q(\underset{\sim}{v})\underset{\sim}{v} = 0$$

(3.1)

$$P(\underset{\sim}{v}) = \tilde{B}_0(\underset{\sim}{v}) \frac{\partial}{\partial t} + \frac{\partial}{\partial t}\left(B_0(\underset{\sim}{v})\cdot\right) , \quad Q(\underset{\sim}{v}) = \sum_{k=1}^{d} Q_k(\underset{\sim}{v}) \equiv \sum_{k=1}^{d} \tilde{B}_k(\underset{\sim}{v}) \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k}\left(\tilde{B}_k(\underset{\sim}{v})\cdot\right)$$

with sufficiently smooth initial data $\underset{\sim}{u}_0(x)$ (at $H^{d/2+1+\varepsilon}$, $\varepsilon>0$, at least) in a time interval $[0,T]$ where a unique smooth solution is a priori known to exist.

Multiplying (3.1) on the left by $\underset{\sim}{v}^*$ we get

$$\frac{\partial}{\partial t}(\underset{\sim}{v}^*\tilde{B}_0\underset{\sim}{v}) + \sum_{k=1}^{d} \frac{\partial}{\partial x_k}(\underset{\sim}{v}^*\tilde{B}_k\underset{\sim}{v}) = 0 . \tag{3.2}$$

To reveal the nature of the conservation law just obtained, we recall the $\underset{\sim}{u}$-$\underset{\sim}{v}$ relation in (2.2), $\frac{1}{\eta_0-1}\cdot\underset{\sim}{u} = \tilde{A}_0\underset{\sim}{v}$, and the $\tilde{B}_0$ notation in (2.5c), $\tilde{B}_0 = \tilde{A}_0$; inserting these we find the first differentiated term in (3.2) is given by

$$\underset{\sim}{v}^*\tilde{B}_0\underset{\sim}{v} = \frac{1}{(\eta_0-1)^2}\cdot\underset{\sim}{u}^*\tilde{A}_0^{-1}\,\tilde{A}_0\,\tilde{A}_0^{-1}\underset{\sim}{u} ; \tag{3.3a}$$

by definition (see (1.3b)), $\tilde{A}_0^{-1} = \frac{\partial\underset{\sim}{v}}{\partial\underset{\sim}{u}} \equiv U_{\underset{\sim}{u}\underset{\sim}{u}}$, and hence by (2.3), the temporal-differentiated term in (3.2) is given by

$$\underset{\sim}{v}^*\tilde{B}_0\underset{\sim}{v} = \frac{1}{(\eta_0-1)^2}\cdot\underset{\sim}{u}^*U_{\underset{\sim}{u}\underset{\sim}{u}}\underset{\sim}{u} = \frac{\eta_0}{\eta_0-1}\,U . \tag{3.3b}$$

Similarily, the spatially-differentiated terms $\underset{\sim}{v}^*\tilde{B}_k\underset{\sim}{v}$ are found to be proportional to the entropy flux components

$$\underset{\sim}{v}^*\tilde{B}_k\underset{\sim}{v} = \frac{\eta_0}{\eta_0-1}\,F^{(k)} , \quad k = 1, 2, \dots d . \tag{3.3c}$$

Thus in (3.2) we restored the conservation of the entropy we have started with, using nothing else but integration by parts

$$(3.4) \qquad \int_{\mathcal{R}^d} U(\cdot,t)d\bar{x} = \int_{\mathcal{R}^d} U(\cdot,t=0)d\bar{x} .$$

Had the equations (3.1) first been linearized and only then integrated by parts we would end up with

$$(3.5a) \qquad \frac{\partial}{\partial t}\left[\int_{\mathcal{R}^d} \underset{\sim}{u}^{*}\tilde{A}_0^{-1}(\bar{x},t)\underset{\sim}{u}\, d\bar{x}\right] = 0 .$$

The conservation of the L_2-type $\tilde{A}_0^{-1}$-norm here is the one induced (due to linearization) by that of the entropy as clearly seen by rewriting (3.4) in the form (apart from the proportional constant $\frac{\eta_0}{\eta_0-1}$)

$$(3.5b) \qquad \frac{\partial}{\partial t}\left[\int_{\mathcal{R}^d} \underset{\sim}{u}^{*}\tilde{A}_0^{-1}(\underset{\sim}{u})\underset{\sim}{u}\, d\bar{x}\right] = 0 .$$

We emphasize that in the latter fully nonlinear case, the renorming via $\tilde{A}_0^{-1}(\underset{\sim}{u})$ depends on the solution itself and hence should <u>not</u> be viewed as an L_2-type (generalized) energy conservation but rather as what it is, namely, an entropy conservation. In fact, by homogeneity <u>the latter asserts conservation in an L_{η_0}-type equivalent quantity.</u>

The above considerations lend themselves to the discrete framework as well. At first stage let us consider only spatial discretization (the method of lines). Introduce a spatial mesh-width $h \equiv \Delta x$ and denote by $\underset{\sim}{w}_\nu(t)$ the approximation at a typical grid point $(\bar{x}_\nu,t)\in\mathcal{R}_h^d\times[0,T]$ where $\bar{x}_\nu = \bar{\nu}h$, $\bar{\nu} = (\nu_1,\ldots,\nu_d)^T$; replacing the spatial differentiation Q in (3.1) by its discrete analogue $h^{-1}Q_h$ based on <u>centered differencing</u> of finite-differences or pseudospectral type (e.g. [7]), we arrive at the differential scheme

$$(3.6) \qquad P(\underset{\sim}{w}_\nu)\underset{\sim}{w}_\nu + h^{-1}Q_h(\underset{\sim}{w}_\nu)\underset{\sim}{w}_\nu = 0 , \quad \bar{x}_\nu\in\mathcal{R}_h^d .$$

Multiplying by $\underset{\sim}{w}_\nu^*$, summing over all grid points and noticing that due to the skew-symmetry of Q_h, summation by parts of the second term on the left vanishes, we conclude (similarily to (3.5))

$$(3.7) \quad \frac{\partial}{\partial t}\left[\sum_{x_\nu \varepsilon \mathscr{R}_h^d} \underset{\sim}{u}_\nu^* \tilde{A}_0^{-1} \underset{\sim}{u}_\nu \Delta x\right] \propto \frac{\partial}{\partial t}\left[\sum_{x_\nu \varepsilon \mathscr{R}_h^d} \underset{\sim}{w}_\nu^* \tilde{B}_0 \underset{\sim}{w}_\nu \Delta x\right] = 0 .$$

As before, while in the linearized case we find an L_2-type discrete energy conservation, it is the discrete entropy which is conserved in the nonlinear problem. Of course, the two may coinside as in the cases of the shallow-water equations (cf. the quasiconservative schemes in [4] and the references therein) and the incompressible Euler equations, where the total energy serves as a generalized entropy function. (In the second <u>singular</u> case the pressure is standardly handled in an independent manner via the poisson equation so that we are left with the convective terms for which the above skew-differencing derivation is indeed effective -- the symmetrizing variables $\underset{\sim}{v}$ are identified with the velocity components and $\tilde{B}_0 = I_3$).

Next we consider time discretization. Introduce a temporal increment Δt and let $\underset{\sim}{w}_\nu^n$ denote the approximation at time level $t = n\cdot\Delta t$. Trying to maintain the quasiconservative property, we focus our attention on time discretization of either the Leap-Frog type [(1)]

$$(3.8) \quad \left[\tilde{B}_0^n (\underset{\sim}{w}_\nu^{n+1} - \underset{\sim}{w}_\nu^{n-1}) + (\tilde{B}_0^{n+1} \underset{\sim}{w}_\nu^{n+1} - \tilde{B}_0^{n-1} \underset{\sim}{w}_\nu^{n-1})\right] + 2\lambda Q_h^n \underset{\sim}{w}_\nu^n = 0, \qquad \lambda \equiv \Delta t/\Delta x ,$$

or the Crank-Nicholson one (here $\underset{\sim}{w}_\nu^{n+\frac{1}{2}} = \frac{1}{2}\cdot(\underset{\sim}{w}_\nu^{n+1} + \underset{\sim}{w}_\nu^n)$)

$$(3.9) \qquad \left[\tilde{B}_0^{n+\frac{1}{2}}(\underset{\sim}{w}_\nu^{n+1} - \underset{\sim}{w}_\nu^n) + (\tilde{B}_0^{n+1}\underset{\sim}{w}_\nu^{n+1} - \tilde{B}_0^n \underset{\sim}{w}_\nu^n)\right] + \lambda Q_h^{n+\frac{1}{2}} \underset{\sim}{w}_\nu^{n+\frac{1}{2}} = 0 .$$

The former has the advantage of being explicit yet nonlinear instability may be excited, cf. [7, Section 5]. This can be remedied by adding a small amount of

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(1) Other variants are also possible, for example $\tilde{B}_0^n(\underset{\sim}{w}_\nu^{n+1} - \underset{\sim}{w}_\nu^{n-1}) + \lambda Q_h^n \underset{\sim}{w}_\nu^n = 0.$

dissipation(1), much smaller than required in the non-skew-selfadjoint case. Turning to the Crank-Nicholson scheme in (3.9), we multiply by $\underset{\sim}{w}^{*n+1}_{\nu}$ and sum by parts the spatial terms ending with an L_2-type energy estimate provided $\tilde{B}_0$ is slowly varying with respect to time(2) (otherwise a discrete entropy conservation does not necessarily follow). Here we note that the use of aligned splitting to overcome the inherent difficulty in solving the multidimensional implicit equations (3.9) is highly desirable -- on account of the skew-symmetric differencing in each spatial direction, the quasiconservative properties will be maintained in each of the splitting substeps and therefore overall.

Our final point concerns the possibility of one-sided differencing of (3.1) where the first and second terms in each of the aligned pairs are forward and backward differenced respectively, so that overall skew-selfadjoint form is maintained. Note that the above quasiconservative schemes (in either the centered or one-sided version) are inappropriate for use under the nonsmooth regime if only because of their conservation of entropy which actually decreases across shocks; small amount of viscosity is required to ensure the irreversibility of our marching procedure, preventing the development of "rarefied shocks". We mention in this respect, one-sided splitting in the spirit of Steger and Warming [11], which is well-accommodated into the (last term of) form (2.5b); the symmetry of the split Jacobians is of importance here.

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(1) For example, replacing $\underset{\sim}{w}^{n\pm1}_{\nu}$ by $[I\mp\varepsilon \sum_{k=1}^{d} (h^2 D_{x_k+} D_{x_k-})^m]\underset{\sim}{w}^{n\pm1}_{\nu}$ would result in (absolute) entropy dissipaton, [6].

(2) This is the case, for example, of the above mentioned incompressible Euler equations where $\tilde{B}_0 \equiv I_3$. Also, when marching toward steady state.

4. Entropy Decay in Bounded Regions

Consider the system (2.5) in the bounded slab $\Omega\times[0,t]$, $\Omega\subset\mathcal{R}^d$, $0 < t < \infty$. The entropy inequality (1.2b) asserts (here, $\bar{F} \equiv (F^{(1)},\dots,F^{(d)})^T$, and the Ω outward normal $\bar{n} \equiv (n_1,\dots,n_d)^T$)

(4.1) $$\int_{\Omega} U(\cdot,t)d\bar{x} \le \int_{\Omega} U(\cdot,t=0)d\bar{x} - \int_{\tau=0}^{t}\left[\int_{\partial\Omega}\bar{F}(\cdot,\tau)\cdot\bar{n}\partial\bar{x}\right]d\tau.$$

Thus, there is a decay in the amount of entropy enclosed provided the entropy outflux requirement is fulfilled

(4.2a) $$\int_{\partial\Omega} \bar{F}(\cdot,t)\cdot\bar{n}\, d\bar{x} \ge 0 .$$

The outflux requirement can also be written as a maximal dissipativity-like condition (see (3.3c))

(4.2b) $$\frac{\eta_0-1}{\eta_0}\cdot\int_{\partial\Omega} \underset{\sim}{v}^*\left[\sum_{k=1}^{d}\tilde{B}_k n_k\right]\underset{\sim}{v}\,\partial\bar{x} \ge 0 ;$$

note the change in sign for $0 < \eta_0 < 1$ -- it can be easily seen that this must be so in order to accommodate the negative sign of U in this case, see (2.3).

A more detailed information can be obtained by taking into account (bounds of) the speed in which entropy propagate. Specifically, we state

THEOREM 4.1. There is a local entropy decay estimate

(4.3) $$\int_{|\bar{x}|\le M} U(\cdot,t)d\bar{x} \le \int_{|\bar{x}|\le M+st} U(\cdot,t=0)d\bar{x} .$$

Here the speed magnitude, $|s|$, is given by the supremum over all $|\bar{\omega}| = 1$, $\bar{\omega} = (\omega_1,\dots\omega_d)^T$, of the largest eigenvalue (in absolute value) of

(4.4a) $$\sum_{k=1}^{d}\frac{\eta_0}{\eta_0+\eta_k-1}\cdot\omega_k A_k(\underset{\sim}{u}) ,$$

with $\underset{\sim}{u}$ varies over all attainable values along the mantle $|\bar{x}| = M + s\ (t-\tau)|_{0 \le \tau \le t}$; the speed orientation is given by

$$\text{(4.4b)} \qquad \operatorname{sgn}(s) = \operatorname{sgn}(\eta_0 - 1)\ .$$

Thus we can distinguish between two cases:

in the first, $0 < \eta_0 < 1$ and hence U attains only negative values. In this case we estimate the spread of entropy decay.

$$\text{(4.5a)} \qquad \int_{|\bar{x}| \le M+|s|\cdot t} U(\cdot,t)\,d\bar{x} \le \int_{|\bar{x}| \le M} U(\cdot,t=0)\,d\bar{x}\ ;$$

otherwise $\eta_0 > 1$, so that U attains only positive values. Here we have

$$\text{(4.5b)} \qquad \int_{|\bar{x}| \le M} U(\cdot,t)\,d\bar{x} \le \int_{|\bar{x}| \le M+|s|\cdot t} U(\cdot,t=0)\,d\bar{x}\ .$$

Estimate (4.5b) may serve as a standard energy estimate over domain of dependence except its being of L_{η_0}-type (due to homogeneity) rather than of L_2-type as usual; it implies, in particular, the existence of finite propagation speed $|s|$, i.e., if $\underset{\sim}{u}_0(\bar{x})$ has compact support inside $|\bar{x}-\bar{x}_0| \le M$, so is $\underset{\sim}{u}(\bar{x},t)$ inside $|\bar{x}-\bar{x}_0| \le M + |s|\cdot t$. In both cases, (4.5a) and (4.5b) provide us with a local entropy decay estimate sharpening the global one we have found in (3.4).

PROOF. Integrating the entropy inequality (1.2b) over the truncated cone $\mathscr{C} = \{(|\bar{x}| \le M + s\cdot(t-\tau) \mid 0 \le \tau \le t\}$, then by Green's theorem for measures [12] we end up with (here $(n_0,\bar{n})$ is $\mathscr{C}$-outward normal)

$$\text{(4.6)} \qquad \int_{\partial\mathscr{C}} \left[Un_0 + \bar{F}\cdot\bar{n}\right]\partial\bar{x} \le 0\ .$$

The integrals over the top and bottom surfaces give us the difference between the left and right hand sides in (4.3); by (4.6), this difference -- which we claim to be nonpositive -- is bounded from above by

$$\text{(4.7a)} \qquad -\int_{\text{mantle}} \left[Un_0 + \bar{F}\cdot\bar{n}\right]\partial\bar{x}\ ;$$

the result follows upon showing that the last quantity is indeed nonpositive. On the mantle n_0 is proportional to s and $\bar{n}$ to $\bar{\omega} \equiv \bar{x}/|\bar{x}|$ and hence using (3.3b) and (3.3c), the integrand in (4.7a) is given by (apart from the positive proportional constant $\frac{1}{\eta_0[s^2+|\bar{\omega}|^2]^{\frac{1}{2}}}$)

$$(4.7b) \qquad Un_0 + \bar{F}\cdot\bar{n} \propto (\eta_0-1)\cdot \underset{\sim}{v}^* \left[s\tilde{B}_0 + \sum_{k=1}^{d} \omega_k \tilde{B}_k \right] \underset{\sim}{v}$$

Thus it is left to verify that $(\eta_0 -1)\cdot[s\tilde{B}_0 + \sum_{k=1}^{d} \omega_k \tilde{B}_k]$ is positive semi-definite. We distinguish between two cases:

(i) $0 < \eta_0 < 1$; in this case we choose negative $s = -|s|$, small enough so that after multiplying both sides of (4.7b) by $\tilde{B}_0^{-\frac{1}{2}}$ we will have

$$(4.8) \qquad sI_N + \tilde{B}_0^{-\frac{1}{2}} \left[\sum_{k=1}^{d} \omega_k \tilde{B}_k \right] B_0^{-\frac{1}{2}} \leq 0 ,$$

i.e., we want the eigenvalues of the second symmetric term on the left hand side of (4.8) to be bounded from above by $|s|$. This is indeed the case since the eigenvalues are exactly those introduced in (4.4a) as follows from the similarity relation, see (2.5c) and (1.3c),

$$(4.9) \qquad \tilde{B}_0^{-\frac{1}{2}} \left[\sum_{k=1}^{d} \omega_k \tilde{B}_k \right] \tilde{B}_0^{-\frac{1}{2}} = \tilde{A}_0^{-\frac{1}{2}} \left[\sum_{k=1}^{d} \frac{\eta_0}{\eta_0+\eta_k-1}\cdot\omega_k A_k \right] \tilde{A}_0^{\frac{1}{2}} .$$

(ii) $\eta_0 > 1$; here we choose positive $s = |s|$ large enough to make the left hand side of (4.8) positive semi-definite. Continuing as before we end up with the same bound for the speed magnitude $|s|$, so that only its orientation is reversed. This completes the proof.

Of particular interest is the case of first order homogeneous fluxes $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$ where $\eta_k = 1$ (we refer again to the example of the aforementioned gasdynamics equations). The propagation speed is bounded in terms of the largest one induced by the 'nonlinear symbol', $\sum_{k=1}^{d} \omega_k A_k$, and naturally this is the best one can hope for viewing it as an extension of the linear theory.

We close this section by making the standard note that a more careful examination of the one-dimensional problem, $d = 1$, will yield sharper speed bounds involving the largest eigenvalue of A_1 on the one hand as well as the smallest one on the other.

5. Non-homogeneous Extensions

The building block for the results obtained in Section 3 & 4 was the skew-selfadjoint form, derived in Section 2 under the homogeneity assumption of both the entropy $U(\underset{\sim}{u})$ and the fluxes $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$. We would like to discuss this assumption and to examine the possibility of deriving skew-selfadjoint form under weaker (nonhomogeneous) conditons.

The entropy, $U(\underset{\sim}{u})$, clearly plays the crucial practical role in deriving the skew-selfadjoint form (2.5) and its η_0-order homogeneity is a much more delicate question than that of the fluxes $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$ as seemed to be indicated, for example, by the clear distinction we have found in Section 4 between the cases $\eta_0 < 1$ and $\eta_0 > 1$. In the gasdynamics equations, it was the freedom we have in choosing any convex function of $S = c_v \ln(p\rho^{-\gamma})$ as a generalized entropy density, which enabled us to meet the homogeneity assumption.

As far as the fluxes $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$ are concerned, their assummed homogeneity is plausible on the ground of dimensional analysis; that is, $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$ are to be in particular first order homogeneous provided no dimensional constants are involved. Included in this specially attractive category (see the end remarks in Sections 2 & 4) are the gasdynamics and slab-symmetrical MHD equations.

In many cases, however, dimensional constants do appear and the homogeneity assumption is not met -- the gravitational constant, g, in the shallow-water equations and the magnetic permeability, μ, in the magnetohydrodynamic equations are just two examples. To this end we observe that the only additional ingredient required for the derivation of (2.5) is the homogeneous dependence of all directional fluxes ($\underset{\sim}{u}$ is included as the temporal one) on the dependent variable $\underset{\sim}{v} = U_{\underset{\sim}{u}}$. The homogeneity requirement therefore can be considerably weakened; for example, assume instead that each of the fluxes can be written as a sum (here, for simplicity we identify $\underset{\sim}{f}^{(0)} \leftrightarrow \underset{\sim}{u}$)

$$(5.1) \qquad \underset{\sim}{f}^{(k)} \sim \sum_{\ell} \underset{\sim}{f}^{(k\ell)}, \qquad k = 0, 1, \ldots, d \quad ,$$

with $\underset{\sim}{f}^{(k\ell)}$ homogeneous of order $\eta_{k\ell}$ in $\underset{\sim}{v}$ (which at least locally is always possible). Then, employing Euler's identity to <u>skew-differentiate each</u> $\underset{\sim}{f}^{(k\ell)}$ as was done above, a final skew-selfadjoint form then follows. More precisely, denote $\tilde{A}_{k\ell} \equiv \frac{\partial}{\partial \underset{\sim}{v}}[\underset{\sim}{f}^{(k\ell)}]$, we write

$$\frac{\partial}{\partial x_k}[\underset{\sim}{f}^{(k)}] = (\sum_{\ell} \theta_{k\ell} \tilde{A}_{k\ell}) \frac{\partial}{\partial x_k}[\underset{\sim}{v}] + \frac{\partial}{\partial x_k}\left[(\sum_{\ell} \frac{1-\theta_{k\ell}}{\eta_{k\ell}} \tilde{A}_{k\ell}) \underset{\sim}{v} \right],$$

where the free parameters $\theta_{k\ell}$ are chosen to make $(\sum_{\ell} \theta_{k\ell} A_{k\ell}) - (\sum_{\ell} \frac{1-\theta_{k\ell}}{\eta_{k\ell}} A^{*}_{k\ell})$ antisymmetric. We recall that $\sum_{\ell} \tilde{A}_{k\ell} \equiv \frac{\partial}{\partial \underset{\sim}{v}}[\underset{\sim}{f}^{(k)}]$ are symmetric; with the further symmetry of <u>each</u> $\tilde{A}^{(1)}_{k\ell}$, the required above choice for $\theta_{k\ell}$ indeed exists, $\theta_{k\ell} = \frac{1}{\eta_{k\ell}+1}$, yielding the skew-selfadjoint form (2.5) with $\tilde{B}_k = \sum_{\ell} \frac{1}{\eta_{k\ell}+1} \tilde{A}_{k\ell}$. Skew-selfadjoint form for the shallow-water equations for example, with

$$(5.2a) \quad \underset{\sim}{u} = (h, \underset{\sim}{m}) \;, \qquad \underset{\sim}{f}^{(k)}(\underset{\sim}{u}) = (m_k, \frac{m_k}{h}\underset{\sim}{m} + \frac{g}{2} h^2 \underset{\sim}{e}^{(k)}), \; k = 1, 2, 3$$

and symmetrizing variables $\underset{\sim}{v} = (gh - |m|^2/2h^2, \underset{\sim}{m}/h)$ (induced by the total energy serving as a generalized entropy function), can be easily worked out according to the above recipe; expanding the 'temporal flux' $\underset{\sim}{u} \equiv \underset{\sim}{u}(\underset{\sim}{v})$ (here, for simplicity, $\underset{\sim}{v} \equiv (v_1, \underset{\sim}{v}_2)$ with $\underset{\sim}{v}_2 \equiv (v_2, v_3, v_4) = \underset{\sim}{m}/h$),

$$g\underset{\sim}{u} = g \cdot [\underset{\sim}{f}^{(01)} + \underset{\sim}{f}^{(02)} + \underset{\sim}{f}^{(03)}] \equiv (v_1, 0_3)^T + (\tfrac{1}{2}|\underset{\sim}{v}_2|^2, v_1\underset{\sim}{v}_2)^T + (0, \tfrac{1}{2}|\underset{\sim}{v}_2|^2\underset{\sim}{v}_2)^T$$

and skew-differentiate <u>each</u> of the terms on the right, we obtain

$$(5.2b) \quad \tilde{B}_0 \equiv \frac{1}{2} \cdot \frac{\partial \underset{\sim}{f}^{(01)}}{\partial \underset{\sim}{v}} + \frac{1}{3} \cdot \frac{\partial \underset{\sim}{f}^{(02)}}{\partial \underset{\sim}{v}} + \frac{1}{4} \cdot \frac{\partial \underset{\sim}{f}^{(03)}}{\partial \underset{\sim}{v}} = \frac{1}{4g} \cdot [(v_1 + \frac{1}{3}|\underset{\sim}{v}_2|^2)^2 + \frac{1}{72} \cdot |\underset{\sim}{v}_2|^2]_{\underset{\sim}{v}\underset{\sim}{v}} \;;$$

---

(1) Expand the potentials $\chi^{(k)}$, $\chi^{(k)}_{v} = \underset{\sim}{f}^{(k)}$, which by the symmetry of $\tilde{A}_k$ exist, $\chi^{(k)} \sim \sum_{\ell} \chi^{(k\ell)}$, then $\tilde{A}_{k\ell} = \chi^{(k\ell)}_{\underset{\sim}{v}\underset{\sim}{v}}$ are indeed symmetric.

the spatial fluxes can be treated similarily, yielding

$$\tilde{B}_k = \frac{1}{2g}\cdot\Big[\frac{1}{3}\cdot v_1^2 v_{k+1} + \frac{1}{4}\cdot v_1 v_{k+1}|\underset{\sim}{v}_2|^2 + \frac{1}{15}\cdot v_{k+1}^3(|\underset{\sim}{v}_2|^2 - \frac{1}{3}\cdot v_{k+1}^2) + \tag{5.2c}$$
$$+\frac{1}{20}\cdot v_{k+1}(|\underset{\sim}{v}_2|^2 - \frac{2}{3}\cdot v_{k+1}^2)^2\Big]_{\underset{\sim}{v}\underset{\sim}{v}}, \quad k = 1, 2, 3 .$$

Finally we remark on the magnetohydrodynamic equations (augmenting the fluid equations (2.1) with the magnetic field $\underset{\sim}{B}$),

$$\underset{\sim}{u} = (\rho, \underset{\sim}{m}, \underset{\sim}{B}, E^*)^T \tag{5.3a}$$

$$\underset{\sim}{f}^{(k)}(\underset{\sim}{u}) = \Big(m_k, \frac{m_k}{\rho}\underset{\sim}{m} + p^*\underset{\sim}{e}^{(k)} - \frac{1}{\mu}B_k\underset{\sim}{B}, \frac{\underset{\sim}{m}}{\rho}\times\underset{\sim}{B}\times\underset{\sim}{e}^{(k)}, (E^*+p^*)\frac{\underset{\sim}{m}}{\rho} - \frac{1}{\mu}\frac{B_k}{\rho}\,\underset{\sim}{m}\cdot\underset{\sim}{B}\Big)^T \tag{5.3b}$$

Here the starred quantities E*, p* correspond to the unstarred ones in (2.1) with the added magnetic pressure $\frac{1}{2\mu}|\underset{\sim}{B}|^2$. The failure of the homogeneity assumption in this case can be easily traced back to the assummed approximation of the magnetic permeability being a constant μ; indeed, viewing μ as a dependent variable, instead, will yield first-order homogeneous fluxes $\underset{\sim}{f}^{(k)}(\underset{\sim}{u})$(1). The author is unaware of any such physically relevant closed system, however. The extended procedure described above for obtaining skew-selfadjoint form is still applicable in this case, though a <u>infinite</u> expansion in the symmetrizing variables $\underset{\sim}{v}$ is required first. As a simpler example for the use of the latter we consider the quasilinear wave equation (cf. [1])

$$\underset{\sim}{u} = (u_1,u_2)^T, \quad \underset{\sim}{f}^{(1)}(\underset{\sim}{u}) = (-u_2, q(u_1))^T, \quad q' < 0,\ q'' > 0, \tag{5.4a}$$

(1) Dimensional arguments show that this dependence is necessarily first-order homogeneous. Also, the momentum and enrgy fluxes should be updated in this case by the additional $-\rho|\underset{\sim}{B}|^2\cdot(\frac{\partial\mu}{\partial\rho})_T\,\delta_{jk}/2\mu^2$ term added to the stress tensor [8, p.142].

with symmetrizing variables $\underset{\sim}{v} = (-q(u_1),u_2)^T$ induced by the entropy $\frac{1}{2}u_2^2 - \int^{u_1} q(w)dw$. While the first-order homogeneous spatial flux causes no difficulties, for the temporal one we need to expand $u_1 \equiv u_1(v_1 = -q) = \sum u_{1\ell}v_1^\ell$ and to skew-differentiate, obtaining the upper left corner element in B_0, $\sum \frac{1}{\ell+1} u_{1\ell}v_1^{\ell-1}$; collecting the other pieces we end up with

$$(5.4b) \qquad \tilde{B}_0 = \mathrm{diag}\left(\frac{1}{v_1^2}\left[v_1 \cdot u_1(v_1) - \int^{u_1(v_1)} u_1(q)dq\right],\ \frac{1}{2}\right),\ \tilde{B}_1 = \mathrm{antidiag}\left(-\frac{1}{2}, -\frac{1}{2}\right) .$$

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